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ON THE COLE – HOPF SUBSTITUTION

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By making use of an arbitrariness in the known Cole-Hopf nonlinear replacement for the Burgers equation, $\phi_t + \phi \cdot \phi_x = \nu\phi_{xx}$, the replacement is generalized in such a way that it makes possible to solve any boundary problems exactly both for $x \geq 0$ and for $0 \leq x \leq l$. Concrete formulae are obtained with a wide choice of $\phi(0,t)$, $\phi_x(0,t)$; $\phi(0,t)$.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

О подстановке Коула – Хопфа

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Использовав произвол в известном нелинейном преобразовании Коула – Хопфа для уравнения Бюргерса, $\phi_t + \phi \cdot \phi_x = \nu\phi_{xx}$, удалось обобщить преобразование так, что появилась возможность точно решать любые краевые задачи как для $x \geq 0$, так и для $0 \leq x \leq l$. Получены конкретные формулы при широком выборе $\phi(0,t)$, $\phi_x(0,t)$; $\phi(0,t)$.

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1. We will proceed from the transport equation in its general form:

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0, \tag{1}$$

where $\rho(x,t)$ is the density per unit length; $q(x,t)$ is the expense of a substance (that depends on a problem under consideration) per unit time; both $\rho(x,t)$ and $q(x,t)$ are assumed to be differentiable functions, though eq.(1) can be generalized to discontinuous functions, as well [1].

Further study of eq. (1) is essentially based on the assumption for $q(x,t)$ being a density functional:

$$q = Q(\rho) \tag{2}$$

that results in the known, generally, nonlinear equation

$$\partial_t \rho + C(\rho)\rho_x = 0, \quad C(\rho) = Q_\rho. \tag{3}$$

That admits general investigation of the behavior of solutions on (x,t)-plane and provides a complete picture of the diagrams of characteristics for nonlinear waves. It can be shown

It is natural to take $q(x,t)$ in the form

$$q(x,t) = Q(\rho) - Re^{-1}\rho_x - \delta_{xx} - \dots \tag{4}$$

where added terms of the diffusion-dispersion type will smear out the front of vertical characteristics where the overturning starts and will lead to the equation

$$\partial_t \rho + C(\rho)\rho_x = Re^{-1}\rho_{xx} + \delta\rho_{xxx}, \quad (Re, \delta) > 0. \tag{5}$$

(In principle, higher-order derivatives can be taken, however, for our purposes, the expansion (4) suffices.) The coefficient of ρ_{xx}, $Re^{-1} = \nu/\vartheta L$, is the inverse Reynolds number and the term with it describes the process of dissipation caused by diffusion (ν is the kinematic viscosity; ϑ is the characteristic velocity and L is the characteristic size). The term $\delta\rho_{xxx}$ describes dispersion. So, it remains to solve the problem with $Q(\rho)$. It can be taken, for example, in the most general form

$$Q(\rho) = \sum_0^n a_k \frac{\rho^k}{k}. \tag{6}$$

However, since even the case $n = 2$ leads to rather a complicated and, generally, unsolvable problem, we take it in the form

$$Q(\rho) = \sum_0^n a_k \frac{\rho^k}{k} \tag{7}$$

which gives, for eq.(5),

$$\partial_t \rho + (\varepsilon\rho + b)\,\rho_x + Re^{-1}\rho_{xx} + \delta\rho_{xxx}. \tag{8}$$

The transformation $\rho' = \rho + b/\varepsilon$ reduces eq.(8) into the «standard» form [2] (the prime will be omitted):

$$\partial_t \rho + \varepsilon\rho\rho_x + Re^{-1}\rho_{xx} + \delta\rho_{xxx}. \tag{9}$$

The nonlinear equation (9) describes both the dissipation and the dispersion. Note that until now, no assumptions were made about the quantities ε, Re^{-1}, δ the amplitude of nonlinearity, viscosity and the dispersion coefficient.

A detailed investigation of the behaviour of nonlinear waves of a small amplitude ε that allows us to start with solving the linear problem and to trace the behavior of nonlinear waves on the interval $t_\varepsilon \sim 1/\varepsilon$ has been carried out in ref. [2]. If in this case we neglect the dissipation, $Re^{-1} \ll \delta$, we arrive at the KdV equation having played a leading part in formulating the inverse-problem method [4]. If the dispersion can be neglected, $Re^{-1} \gg \delta$, we obtain the Burgers equation [5]

$$\partial_t \rho + \varepsilon\rho\rho_x = Re^{-1}\rho_{xx} \tag{10}$$

that will be further of our detailed investigation.

2. We can apply the scale transformation

$$t = a\tau, \quad x = bz; \quad a = (Re^{-1})^{1/2}\varepsilon^{-1}, \quad b = \sqrt{Re^{-1}}$$

to eliminate the parameters ε and Re^{-1} from equation (10), as a result of which equation (10) reduces to the equation

$$\partial_\tau \rho + \rho\rho_x = \rho_{xx}, \tag{11}$$

where for convenience of comparing with earlier results [1] we omit the parameter ε, i.e., the scale of the amplitude will be $\rho'(x,t)$. Passing from t to τ by the formula $a\tau$, we arrive at the following form of the Burgers equation

$$\partial_\tau \rho + \rho\rho_x = \nu\rho_{xx}, \quad \nu = \frac{Re^{-1}}{\varepsilon}, \quad a = \frac{1}{\varepsilon}. \tag{12}$$

For this equation, Cole [6] and Hopf [7] have independently proposed the nonlinear replacement of variables that reduces (12) to the linear equation of thermal conductivity, namely,

$$\rho = -2\nu \frac{\partial}{\partial x}[\ln \varphi(x,t)]. \tag{13}$$

It is convenient to make this substitution in two steps

$$\rho = \frac{\partial}{\partial x}\psi(x,t). \tag{14}$$

Note that

$$\psi(x,t) = \int_0^x \rho(x,t) + \rho_0(t) \tag{15}$$

obeys equation (14), which was nowhere mentioned. Substitution of (14) into (12) gives

$$\psi_{\tau x} + \psi_x\psi_{xx} - \nu\psi_{xxx} = 0 \tag{16}$$

or

$$\left(\psi_\tau + \frac{1}{2}\psi_x^2 - \nu\psi_{xx}\right)_x = 0. \tag{17}$$

The latter equation is integrated trivially and, as a result, we obtain

$$\psi_\tau + \frac{1}{2}\psi_x^2 - \nu\psi_{xx} = f(\tau), \tag{18}$$

where $f(\tau)$ is a function of τ. Note that in the standard consideration [1,2] it is assumed to be zero. Below, we shall explain for which problems it is possible. Further, putting

$$\psi = -2\nu \ln \varphi, \quad \varphi = \exp\left(-\frac{1}{2\nu}\psi\right), \tag{19}$$

and computing the derivatives in (18), we have

$$\varphi_\tau = \nu\varphi_{xx} - \frac{1}{2\nu}f(\tau)\varphi. \qquad (20)$$

The nonlinear Burgers equation (12) is reduced to the linear equation of thermal conductivity (diffusion) and there appears an extra term with the function $f(\tau)$ given on the characteristic of an equation of the parabolic type.If we consider this substitution as a method of reduction of the nonlinear equation to the equation of thermal conductivity, then in particular problems [1] at $f(\tau)=0$ we can solve only the initial problem, defining $\rho(x,0)=F(x)$ (a physical quantity is just $\rho(x,t)$ entering into (12)) and using the source function for (20), we can obtain a rather complicated expression for $\rho(x,t)$, which is just made in [1,2].

However, we think that the extra term with $f(\tau)$ in (20) will allow us to solve realistic boundary value problem for (8) and (9). We can, of course, formulate the boundary value problems for (20) at $f(\tau)=0$ of the first kind $\varphi(0,t)=\varphi_0(t)$ and of the second kind $\varphi_x(0,t)=\varphi_q(t)$, and to employ the tested methods [8,9,10]. Since the function $\psi(x,t)$ in (14) and the function $\varphi(x,t)$ in (19) are auxiliary functions, we should start with the conditions on the real density obeying (11), i.e., formulate both the initial and boundary value conditions, defining $\rho(0,t)=\rho_0(t)$, $\rho(x,0)=\rho_0(x)$, $\rho_x(0,t)=\rho_1(t)$ with various combinations of boundary value conditions [8,9] and making there the Cole — Hopf substitution, solve (20) with the functions thus defined rather than with an arbitrary function $f(\tau)$.

This program will be realized below. When (12) and (20) are used in concrete problems, we should take into account that we removed the constant in (8), passing from the density ρ in (1) to $\rho'=\rho+b/\varepsilon$, events evolved in time $\tau=\varepsilon t$, and the diffusion coefficient in (12) was equal to $\nu=Re^{-1}\varepsilon$. The Reynolds coefficient Re and the amplitude of nonlinearity are both nonzero, but they can, in principle, possess any values.

3. Now we shall take advantage of those possibilities that arise owing to the functional arbitrariness (18) for solving eq.(12) in the region $t\geq 0, 0\leq x<\infty$. Using (15) we obtain

$$f(\tau) = \lim_{x\to 0}\left[\psi_\tau(x,t) + \frac{1}{2}\psi_x^2(x;t) - \tau(\psi_x)_x\right] = \rho'_0(\tau) + \frac{1}{2}\rho_0^2(\tau) - \nu\rho_x(\tau), \qquad (21)$$

where

$$\rho_0(\tau) = \rho(x=0,\tau), \qquad \rho_x(\tau) = \rho_x(0,\tau) \qquad (22)$$

are known functions given as boundary conditions of the Burgers equation (12). In addition, to find solutions of eq.(12) in the region $0\leq x<\infty$, $t\geq 0$, one should define $\rho_0(x)=\rho(x,\tau=0)$. The spatial coordinates and time coordinates in the integrals written below will be denoted by (x,ξ,z) and (τ,μ,λ), respectively. Using the substitution

$$\varphi = \chi(\tau)R(x,\tau) \qquad (23)$$

it can be shown that the function $R(x,\tau)$ obeys the equation

$$R_\tau(x,\tau) = \nu R_{xx}(x,\tau) \tag{24}$$

provided that

$$\chi = \exp\left(-\frac{1}{2\nu} \int_0^\tau f(\mu) d\mu \right) = \chi_0 e^{\frac{1}{2}\rho_0(0)} e^{-\tau}$$

$$r = \frac{1}{2\tau}\left[\rho_0(\tau) + \int_0^\tau \left(\frac{1}{2}\rho_0^2(\mu) - \nu\rho_1(\mu) \right) d\mu \right] \tag{25}$$

and χ_0, $\rho_0(0)$ are taken from physical considerations. In view of relation (19) we obtain

$$R(x,\tau) = R_0^{-1} \exp\left(r - \frac{1}{2\nu}\psi(x,\tau) \right) \equiv$$

$$\equiv R_0^{-1} \exp \frac{1}{2\nu}\left[\int_0^\tau \left(\frac{1}{2}\rho_0^2(\mu) - \nu\rho_1(\mu) \right) d\mu - \int_0^x \rho(\chi,\tau) d\chi \right]$$

$$R_0 = \chi_0 \, e^{\frac{1}{2}\rho_0(0)} \tag{26}$$

using (15) and (19). From expression (19) it follows that

$$R_1(x) \equiv R(x,0) = R_0^{-1} \exp\left[-\frac{1}{2\nu} \int_0^x \rho(\xi, 0) d\xi \right]$$

$$R_2(x) \equiv R(0,\tau) = R_0^{-1} \exp \frac{1}{2\nu}\left[\int_0^\tau \left(\frac{1}{2}\rho_0^2(\mu) - \nu\rho_1(\mu) \right) d\mu \right].$$

So, we completely define the first boundary value problem [3,9] for an auxiliary equation of the diffusion type (24). It is not difficult to formulate the second and third boundary value problems for eq.(24) (along with $R_1(x)$, either $R_3(\tau) \equiv \partial_x R(0,\tau)$ or $\partial_x R(0,\tau) = \lambda[R(0,\tau) - \theta(\tau)]$ is given), as well as the problem without initial conditions $((R_1(x) = 0))$ and various combinations of boundary value problems. One can also solve problems in the interval $0 \le x \le l$ [3]. These problems require the definition of $R(l,\tau)$, $R_x(l,\tau)$, or $\rho_x(l,\tau)$, (the latter will result in more complicated formulae but difficulties are of pure computational nature).

The first term in (21) can be expreseed through $\rho_0(x)$, $(\rho_0)_x$, $(\rho_0)_{xx}$ by using eq.(12)

$$\rho'_0(\tau) = -\rho_0(\tau)\rho_x(\tau) + \nu\rho_{xx}(0,\tau) \tag{27}$$

but keeping in mind (23) with further representation (25) for $\chi(\tau)$ we have not made that.

The first boundary value problem for (24) is solved by the conventional method [3,9]. Its solution is given by the formulae

$$R(x, \tau) = R_1(x, \tau) + R_2(x, \tau); \tag{28}$$

$$R(x, \tau) = \int_0^\infty [G_-(x, \xi, \tau) - G_+(x, \xi, \tau)] R_1(\xi) d\xi +$$

$$+ 2\nu \int_0^\tau \frac{\partial G(x, 0, \tau - \lambda)}{\partial \xi} R_2(\lambda) d\lambda$$

$$R_1(\xi) = R_0^{-1} \exp\left(-\frac{1}{2\nu} \int_0^\xi \rho_0(\xi) d\xi\right)$$

$$R_2(\lambda) = R_0^{-1} \exp\left[-\frac{1}{2\nu} \int_0^\lambda \left(\nu\rho_1(\mu) - \frac{1}{2}\rho_0^2(\mu)\right) d\mu\right]$$

$$G_\pm(x, \xi, \tau) = \frac{1}{\sqrt{2\pi d}} \exp\left(-\frac{h_\pm^2}{2d}\right), \quad h_\pm = x_\pm \xi, \quad d = 2\nu\tau$$

$$\frac{\partial G(x, 0, \tau - \lambda)}{\partial \xi} = -\frac{2\pi x}{(2\pi d_1)^{3/2}} \exp\left(-\frac{x^2}{2d_1}\right), \quad d_1 = \nu(\tau - \lambda).$$

The solution $\rho(x, \tau)$ for the Burgers equation can be derived on the basis of formulae (14), (19), (23) and is given by the formula

$$\rho(x, \tau) = \psi_x = -2\nu \frac{R_x}{R}, \tag{29}$$

where

$$R_x = \int_0^\infty \left[\frac{\partial G_-(x, \xi, \tau)}{\partial x} - \frac{\partial G_+(x, \xi, \tau)}{\partial x}\right] R_1(\xi) d\xi +$$

$$+ 2\nu \int_0^\tau \frac{\partial G_{-0}(x, 0, \tau - \lambda)}{\partial x \partial \xi} R_2(\lambda) \partial\lambda$$

$$\partial_x G_\mp(x, \xi, \tau) = \pm \frac{2\pi h_\mp}{(2\pi d)^{3/2}} \exp\left(-\frac{h_\mp^2}{2d}\right)$$

$$\partial_{x\xi} G(x, 0, \tau - \lambda) = -\left[2\pi - \frac{(2\pi x)^2}{2\pi d_1}\right] \exp\left(-\frac{x^2}{2d_1}\right) (2\pi d_1)^{-3/2}. \tag{30}$$

Naturally, the final formulae for $\rho(x, \tau)$ obtained by (29) are rather complicated. However, defining the initial density distribution $\rho_0(x) = \rho(x, \tau = 0)$ obeying the burgers equation and the values of density and its first derivative $\rho_0(\tau) = \rho(x = 0, \tau)$, $\rho_1(\tau) = \rho_x(x = 0, \tau)$, we obtain the evolution of $\rho(x, \tau)$ law for $\tau \geq 0$ and $0 \leq x \leq \infty$ with the help of (29) and (30). The functions $\rho_0(x)$, $\rho_0(\tau)$, $\rho_1(\tau)$ should satisfy the conditions

$$|\rho_0(x)| < A, \quad |\rho_0(\tau)| < B, \quad |\rho_1(\tau)| < C$$

that ensure the obtained solution being unique.

Now, we can formulate the following problems:

1. the first boundary value problem for the Burgers equation defining $\rho_0(x) = \rho(x, \tau = 0)$; $\rho(x = 0, \tau) = \rho_0(\tau)$, $\rho_x(0, \tau) = 0$;
2. the second boundary value problem for the same equation giving $\rho_0(x)$, $\rho_x(0, \tau) = \rho_0(\tau)$ and [3,9];
3. the third boundary value problem defining $\rho_1(\tau) = \lambda[\rho_0(\tau) - \theta(\tau)]$.

Formulae (29) and (30) give their solution.

Thus, for describing a specific process, it is necessary to replace ρ, ν, by , $\rho - b/\varepsilon$, $\nu = Re^{-1}/\varepsilon$, $t = \tau/\varepsilon$, $d = \nu\tau \to Re^{-1}$ in $R_1(\xi)$, $R_1(\lambda)$ and $R(x, \tau)$.

In conclusion, we stress once more that when (12) and (20) are used in concrete problems, it is necessary to take into account that we omitted the constant from (8) passing from the density ρ to $\rho' = \rho + b/\varepsilon$ in eq.(1), processes evolved with time $\tau = \varepsilon t$, and the diffusion coefficient was $\nu = Re^{-1}/\varepsilon$ in eq.(12). The Reynolds number and the nonlinearity coefficient are both different from zero but can assume any values.

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